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Research article

Relationships between domain changes and connections: An analysis from the perspective of mathematical work and teacher knowledge

Relaciones entre cambios de dominio y conexiones: Análisis desde la perspectiva del trabajo matemático y conocimiento del profesor

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Abstract:
Introduction: This study addresses the teaching task contextualized in the teaching of different topics of mathematic (functions, sequences, and Thales' theorem). To do so, the characteristics of the mathematical work and specialized knowledge that the teacher shows during teaching are analyzed, considering as a frame of reference the complementarity between two theoretical perspectives (MWS and MTSK). **Methodology:** The study is framed in the qualitative approach, based on an integrated multiple case study design, which allows analysis of the proposed mathematical performance (oral and/or written) of three teachers. **Results:** The analyses allow a deep examination of the development of mathematical work and demonstration of the relationships between the inter-conceptual connections as well as the domain changes that are expressed in teachers' work. In general, the results reveal the privileging of algebraic treatments and emphasis on the procedural. **Conclusion:** Contributions are identified in the perspective of the connection between theories in the context of the models involved.

Keywords: mathematical teacher's specialized knowledge; mathematical working spaces; connections between theories; domain changes; Thales' theorem; sequences; concept of the function; teaching of mathematics.

Resumen:
Introducción: El presente estudio aborda el quehacer docente contextualizado en la enseñanza de diferentes temas matemáticos (funciones, sucesiones y teorema de Tales). Para ello, se analizan las características del trabajo matemático y del conocimiento especializado que el profesor muestra durante la enseñanza, considerando como marco de referencia la complementariedad entre dos perspectivas teóricas (ETM y MTSK). **Metodología**: Se enmarca en el enfoque cualitativo, basado en un diseño de caso múltiple de tipo integrado, lo que permite analizar el desempeño matemático propuesto (oral y/o escrito) de tres profesores. **Resultados**: Los análisis permiten ahondar en el desarrollo del trabajo matemático y evidenciar las relaciones entre las conexiones inter-conceptuales junto a los cambios de dominio que se expresan en dicho trabajo. De modo general, los resultados revelan el privilegio de tratamientos algebraicos y énfasis en lo procedimental. **Conclusión:** Se identifican aportes en la perspectiva de la conexión entre teorías en el contexto de los modelos involucrados.

Palabras clave: conocimiento especializado del profesor; espacios de trabajo matemático; conexiones entre teorías; cambios de dominio; teorema de Tales; sucesiones; concepto de función; enseñanza de la matemática.

1. Introduction

Attempting to understand the teacher's practice allows for the development of studies with various orientations: economic, cultural, social, psychological, educational, and others. This article addresses the task of the teacher contextualized in the teaching of different mathematical topics, with a viewpoint narrowed to two foci: the mathematical work proposed in the classroom and the knowledge mobilized to that end

The concern for this topic has its origins in the work of Shulman (1986), who places emphasis on the knowledge specialization of the teacher in the discipline that they teach. In addition to the latter is added the importance of the role of the teacher and their knowledge regarding the interactions that take place in the classroom, particularly those that occur in the context of mathematical work, as illustrated by Flores-Medrano et al. (2016) and Gómez-Chacón et al. (2016).

This study examines the work proposed by the teacher, focusing on the connections between their knowledge and the changes of mathematical topic or domain produced by the teacher, for which two theoretical approaches will be utilized: Mathematics Teacher's Specialised Knowledge (MTSK) (Carrillo et al., 2018) and Mathematical Working Spaces (MWS) (Kuzniak et al., 2022). Both approaches will be applied to the analysis of the teaching of three topics in the Chilean national curriculum: functions, sequences in R, and Thales' theorem (TT). The selection of these topics lies in their importance within the development of mathematical knowledge and the fact that they allow the establishment of connections with diverse areas, underscoring their complexity and the diversity of approaches in their study. This work builds on previous studies that use both approaches to study these topics and allow accounting for the relations that this article intends to elucidate (Climent et al., 2021; Espinoza-Vásquez & Verdugo-Hernández, 2022; Espinoza-Vásquez et al., 2022; Henríquez-Rivas et al., 2021; Verdugo-Hernández et al., 2022).

Meanwhile, the use of these two theoretical approaches to analyze the same dataset is framed within the paradigm of connection between theories (Prediger et al., 2008), which establishes different strategies to characterize said connection. Along with this, Radford (2008) indicates three characteristics of theories which, in turn, guide the establishment of such connections. Specifically, Radford states that a theory is characterized by its Principles (P), Methodologies (M), and Pragmatic Research Questions (Q). These elements are identifiable both in the theory of Mathematical Working Spaces (MWS) and in the conceptualization of Mathematics Teacher's Specialised Knowledge (MTSK), as exemplified by Verdugo-Hernández et al. (2022), emphasizing that it is possible to discuss the connection between the two theories in question.

Although an increase in the number of studies using more than one theory has been observed in the past several years (Castella, 2021), the MWS-MTSK relationship is comparatively recent, and has proved fruitful in allowing for new interpretations of what each theoretical perspective can achieve when being applied separately and to understand what the teacher does in light of what they know. For example, the literature review carried out by Espinoza-Vásquez et al. (2022) demonstrates the connection strategies used for different works and what the main contributions have been in establishing the link in question. Said review highlights the complementarity between the two theories and their mutual capacity to refine respective analyses.

Considering the latter, this article focuses on the connections between different areas of the mathematics teacher's knowledge and the domain changes that they effect associated with the teaching of specific topics, including functions, sequences, and Thales' theorem. To this end, the objective of the study is to characterize the teacher's mathematical work and specialized knowledge mobilized during the teaching of these topics, as well as the domain changes that occur in the process.

2. Theoretical framework

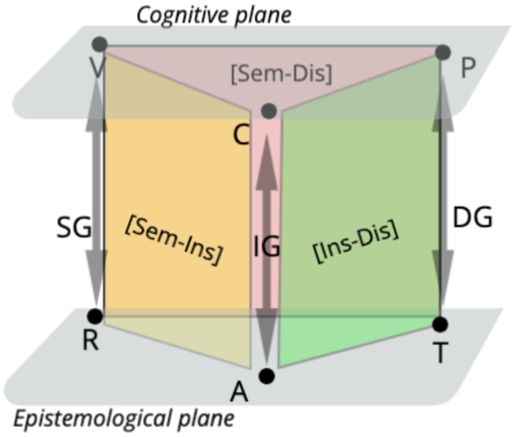
This section presents the formulation of the theoretical perspectives in question: *Mathematical Working Spaces* (MWS) and *Mathematics Teacher's Specialised Knowledge* (MTSK), their components, and existing literature on their complementarity.

To begin with, Mathematical Working Spaces (MWS) is a theory that allows the work of an individual to be analyzed through a given task (Gómez-Chacón et al., 2016; Kuzniak & Nechache, 2021; Kuzniak et al., 2022). This theory presents two planes, epistemological and cognitive, which are linked via semiotic, instrumental, and discursive geneses.

In MWS, the plane related to an individual's thinking—the cognitive plane—is comprised of three interacting components: Visualization, Construction, and Proof. Likewise, the plane related to the mathematical content of a given topic—the epistemological plane—consists of the following components: Representamen, Artefacts, and Theoretical Referential. The Semiotic, Instrumental, and Discursive geneses link the respective planes via these components (Figure 1).

Figure 1.

MWS planes and geneses



Source: MWS model adapted from Kuzniak et al. (2022).
Note. V: Visualization; C: Construction; P: Proof; R: Representamen; A: Artefact; T: Theoretical Referential. SG: Semiotic Genesis; IG: Instrumental Genesis; DG: Discursive Genesis.

Semiotic genesis connects the representamen with the process of visualization. Meanwhile, instrumental genesis links the artefact component with processes of construction, and finally, discursive genesis joins the referential with the discursive reasoning of the proof. Further, the activation and interaction between two of these geneses gives way to three vertical planes that help specify the trajectories of the mathematical work based on which geneses are activated: semiotic-instrumental [Sem-Ins], semiotic-discursive [Sem-Dis], and instrumental-discursive [Ins-Dis]. The activation of these geneses and planes occurs through work on a mathematical task, whose development permits the interpretation of the mathematical work trajectories of an individual (teacher, student).

Together with the above, the epistemological plane includes three types of tools: theoretical, technological, and semiotic, associated with the Theoretical Referential, Artefact, and Representamen components, respectively. The theoretical tool refers to reasoning based on logic and the properties of mathematical objects which pertain to the referential of the task being solved. Technological tools correspond to artefacts such as drawing tools or routine techniques based on algorithms or calculators with calculation algorithms. Finally, semiotic tools are non-material tools used to operate on semiotic representations of mathematical objects (Kuzniak et al., 2016). In addition to this ensemble of tools, operational tools are also recognized, referring to those theoretical tools that are utilized to solve a given task, but which do not form part of the theoretical referential to which said task belongs (Verdugo-Hernández, 2018).

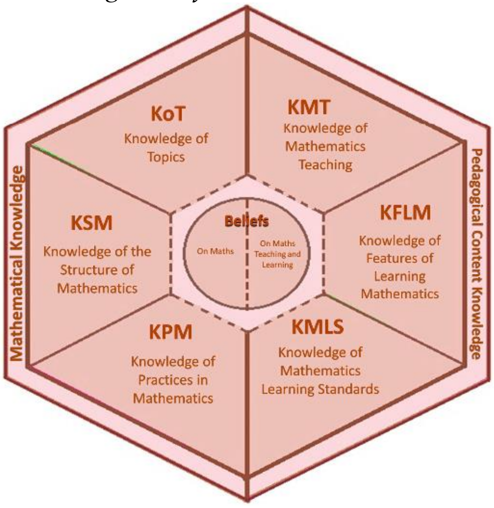
Meanwhile, MTSK is presented as an analytical model of the mathematics teacher's knowledge and a tool for the analysis of their practices (Carrillo et al., 2014). It is composed of three domains: Mathematical Knowledge (MK), Pedagogical Content Knowledge (PCK), and the domain of beliefs about mathematics and its teaching-learning (Figure 2). According to

Carrillo et al. (2018), these beliefs permeate the other two domains, which can impact the manner in which knowledge is manifested.

In the MTSK, knowledge of mathematics as a scientific discipline and body of knowledge is considered within the domain of Mathematical Knowledge (MK), which includes the following: knowledge of mathematical topics (KoT) and their conceptual networks, definitions, theorems, notations, representations, and procedures; knowledge of the structure of mathematics (KSM), addressing the connections between concepts that can help to work on a given topic (auxiliaries), to recognize qualities in common (transversals), or to show the conceptual evolution of a topic (simplification or complexity); and lastly, knowledge of practices in mathematics (KPM), viewed as knowledge regarding the manner in which mathematical ideas are produced, explored, and communicated (left side of Figure 2).

Figure 2.

Schematization of subdomains and categories of the mathematics teacher's specialized knowledge



Source: Carrillo et al. (2018, p. 241). https://doi.org/10.1080/14794802.2018.1479981

Meanwhile, the domain of Pedagogical Content Knowledge (PCK), which relates to mathematical topics as objects of teaching and learning, considers knowledge of strategies, examples, resources, and theories for teaching as knowledge of mathematics teaching (KMT). Meanwhile, knowledge of the characteristics of learning mathematics, including the way in which students learn or approach a topic, theories of teaching, and the emotional aspects of learning are categorized as knowledge of features of learning mathematics (KFLM). Lastly, knowledge of mathematics learning standards (KMLS) is understood as that which is stipulated for students to learn along with the depth of this learning at a given grade level (right side of Figure 2).

The specialized character of knowledge lies in specialization for teaching mathematics and its formation as an organic whole, considering the context of the teaching and epistemology of topics (Scheiner et al., 2019). Therefore, the subdivision into domains and subdomains is only for analytical purposes, since it is expected that this knowledge will be presented in an integrated manner in the teacher's practice, resulting from the diverse manners of knowing mathematical content and applying it to the teaching process.

Regarding the above, it should be underscored that even where differences exist in the orientations of each theoretical model, their formulations include elements that constitute meeting points for addressing their connection. For example, both explicitly consider the semiotic perspective regarding mathematical objects, definitions, and properties, as well as the knowledge of use of certain resources that enhance teaching. Additionally, both models place a central role on mathematics from an epistemological point of view and as a source of objects for teaching and learning that inspire a given activity. In this sense, both theoretical perspectives recognize mathematical domains (identified as Topics in the MTSK). In line with Gómez-Chacón et al., (2016), the differentiation between these domains (geometry, algebra, probability, etc.) is related to the type of objects they comprise and their epistemological aspects. Regarding *domain changes*, these are understood as transitions or connections between distinct mathematical domains, one of origin and the other of resolution (Montoya & Vivier, 2014). A domain change can signify work that does not include a return to the domain of origin or a demand for change attributed to the mathematical task for its successful resolution.

The task and the teacher stand out as two facilitating elements in the relationship between MWS and MTSK, whose analysis allows for investigation of the connection between the models (Espinoza-Vásquez et al., 2022; Verdugo-Hernández et al., 2022). Although there is no consensus regarding the definition of *task*, this article adopts the definition used by Henríquez-Rivas et al. (2021): "a mathematical experience planned for students, which can be an action or sequence of actions" (p. 127). Meanwhile, in terms of focusing attention on the teacher, their role is foundational in MTSK, while in MWS it is addressed mainly through the study of one type of MWS: *idoine* or *suitable* MWS (Henríquez Rivas et al., 2022), which is understood as the way in which mathematical contents are designed, adapted, and presented for teaching in a given institution and context (Flores-González & Montoya-Delgadillo, 2016; Kuzniak et al., 2016). In this manner, both perspectives can be connected and implemented through the inherent links they present, or in the search for new connections between their components through the study of teaching practice.

The works of Espinoza-Vásquez and Verdugo-Hernández (2022) and Verdugo-Hernández et al. (2022) explore and identify relations between MWS tools and the KSM and KoT as specialized knowledge mobilized through their usage. This has allowed for explanation in greater detail of the mathematical work of the teacher in light of their mathematical knowledge and the role that they grant this knowledge during their teaching work.

3. Methodology

With the objective of characterizing the relations between domain changes in specific mathematical topics in the context of the mathematical work and knowledge utilized by the teacher during classroom teaching, this study is framed in a qualitative approach. An integrated multiple case study design was utilized (Yin, 2009), with the mathematical performance presented (and/or written) by three teachers constituting the units of analysis. The first case focuses on a secondary school teacher (P1) teaching Thales' theorem (of intersection; or, Strahlensatz theorem) in the context of an continuing educational experience; the second case centers on teaching functions at the secondary level by a second teacher (P2); and the third case addresses teaching sequences at the university level in a calculus course by a professor (P3). These three cases were studied from the MWS and MTSK theoretical perspectives (Climent et al., 2021; Espinoza-Vásquez & Verdugo-Hernández, 2022; Henríquez-Rivas et al., 2021; Verdugo-Hernández et al., 2022), which permits exploration of domain changes and theoretical connections, as well as delving into their complementarities.

The case study is appropriate for this work since it analyzes events developed in real, natural, and particular school contexts (Yin, 2009). Additionally, the objective is to provide an in-depth theoretical analysis. The justification for the integrated multiple case study design is based on the fact that this study addresses a phenomenon in diverse contexts that are first analyzed individually. Thus, the joint analysis of the cases then permits contributing to the theoretical complementarity between the two models being used.

The criteria for the selection of the three cases was based on the fact that they were representative and revealing in terms of the teaching of the mathematical contents involved (Yin, 2009). All participants were considered to be apt informants, considering, as signaled by Loughran et al. (2008), that all were committed to teaching and to the research in which they were invited to participate. Case selection also followed a common criterion of convenience (Creswell, 2014) related to readily accessible informants and data under the conditions of each investigation, as well as each case presenting particular conditions that allowed the obtention of data rich in specialized knowledge and the activation of MWS components. The pseudonyms P1, P2, and P3 are used to refer to the three cases, respectively, guaranteeing the confidentiality of the identities of the informants and participating institutions.

Specifically, P1 is the representative of a group of secondary teachers that selected Thales' theorem as a typical topic of the Chilean national curriculum. P1 emerged as a teacher accustomed to expository and traditional instruction and who possesses ample experience in secondary-level teaching; he is not habituated to using geometric software. Meanwhile, P2 is a secondary mathematics teacher with a master's degree in mathematics, and he has teaching experience in both K-12 and university settings, where he habitually teaches the topic of functions. P3 is a university professor with a doctorate in mathematics, and he is an active researcher in the area. During the study, P3 was teaching a course in integral calculus (Calculus II), a subject including the study of sequences and limits that he has taught for at least five years. All participants took part voluntarily in this study, without any economic compensation and without assuming any personal risk. Moreover, data collection took place in environments adequate for both the communication and confidentiality of each participant.

3.1. Context of each case

In the first case (P1), mathematical work was developed in the context of a workshop for secondary in-service teachers involving sessions in the university and the school along with the teachers' work in their classrooms, in real contexts with students. The workshop was composed of three principal instances. The first included the presentation of tasks that P1 typically presents for teaching Thales' theorem. The second considered discussion and classroom planning of a reformulated task. In the third instance, this reformulated task was implemented in the classroom; from this implementation, a classroom episode was selected for analysis for the first case. P1 stood out among the other teachers by incorporating new elements into his task and considering the use of GeoGebra software. Additionally, he demonstrated availability, commitment, and expressiveness, all of which were considered for his selection in this case.

The case of P2 was observed during classes imparted to a course of Chilean first-year secondary students (aged 14-15) aimed at teaching the function. The content of these classes ranged from the definition of the concept to the presentation of linear and affine functions, the first types of function studied in the Chilean curriculum (Ministerio de Educación [MINEDUC], 2015). For this case, one of P2's classes was selected in which he taught the calculation of images and pre-images of a function through the solving of equations and constructed the graphical representation of a function.

In the case of P3, the presentation of a mathematical task was observed in the context of a summative evaluation and expert execution by P3, who facilitated materials designed for the course, including exercise guides, solved exercises, evaluation instruments developed by him, and the answers developed as expert responses. For the analyses presented, the first part of a test question on sequences and their developments was selected, the objective of which was to conclude with the convergence of a sequence.

3.2. Data collection and analysis

The data originates from research led by the authors of this article. Data collection and analysis is synthesized in Table 1, which shows the principal aspects regarding each case. Content analysis was applied to the data (Bardín, 1996), underpinned by theoretical elements of the MWS and MTSK models and following the protocol shown in Table 2 and Table 3.

Table 1.

Informants, data collection, and analysis

Informant	Data source	Data analysis
P1	Video - recording of a class and its transcription	The analysis of P1's data is based on criteria and components described according to MWS and MTSK theoretical perspectives, utilized separately.
P2	Video - recording of classes and their transcription	In this case, analysis considers integrated MWS and MTSK theoretical perspectives, allowing establishment of relations between the models and of theoretical complementarity focused on domain changes and inter-conceptual connections.
P3	Material designed for evaluation in the course, transcriptions, and labeling of the development	Analysis in this case entails integrated MWS and MTSK theoretical perspectives, focused on the KSM and the use of theoretical, semiotic, and operational tools from MWS, which allows us to delve into the theoretical complementarity of domain changes and inter-conceptual connections.

Source: Own elaboration (2024).

The analysis utilized a deductive approach (Goetz & LeCompte, 1988) in relation to the place of the theoretical aspects in the research. To this effect, the protocol includes descriptors related to the elements comprising each theoretical perspective. The information obtained allowed recognition and discussion of the connections between different knowledge and of domain changes produced in each case. The geneses and/or vertical planes activated in the mathematical work were recognized by using the protocol for analysis according to the MWS model (Table 2).

Table 2.

Protocol for analysis based on criteria, components, and descriptors according to MWS model

Model	Criterion	Components	Descriptor
MWS	Semiotic genesis	Visualization	Processes of interpretation and relation of mathematical objects with cognitive activities linked to registers of semiotic representations.
		Representamen	Relates mathematical objects with their signifying elements or units. Semiotic tool.
	Instrumental genesis	Construction	Based on processes that stem from actions when utilizing artefacts following available usage techniques.

		Artefacts	Considers artefacts that can be material, symbolic systems, or theoretical. Technological tools.
	Discursive genesis	Proof	Processes based on discursive reasoning in a proof to justify, argue, or demonstrate.
		Referential	Considers definitions, properties, or theorems. Theoretical tools.

Source: Own elaboration (2024).

Together with this, the specialized knowledge was identified through the categories included in MTSK model. Table 3 contains the protocol for analysis based on the MTSK model.

Table 3.

Protocol for analysis based on criteria, components, and descriptors according to MTSK model

Model	Criterion	Components	Descriptor
MTSK	Mathematical knowledge	Of the topic	Corresponds to knowledge of properties, definitions, representations, and procedures associated with the topic.
		Of the structure of mathematics	Associated with knowledge of types of connections between mathematical topics.
		Of practices in mathematics	Related to knowledge of practices associated with the exploration, communication, and production of mathematical knowledge in a topic or in mathematics in general.
	Pedagogical content knowledge	Of mathematics teaching	Corresponds to the knowledge of resources, strategies, activities, examples, and theories of teaching that are conditioned by the mathematical topic to be taught.
		Of the features of learning mathematics	Associated with knowledge of theories on learning, difficulties, errors, and the relation between students and a topic.
		Of mathematics learning standards	Corresponds to knowledge of the learning expected at a determined grade level, the depth of learning and the curricular progression of topics.

Source: Own elaboration (2024).

To validate the findings and provide consistency to the results exhibited, this study utilized triangulation of expert researchers (Arias, 2000). Thus, the analysis process for each case was carried out by the three researchers (authors), each a specialist in one of the two theories in question and with experience in their joint usage. Once consensus was attained regarding the findings focused on knowledge, mathematical work, and domain changes, these factors were considered in the formation of the results.

4. Results

This section presents the findings in each of the three case studies.

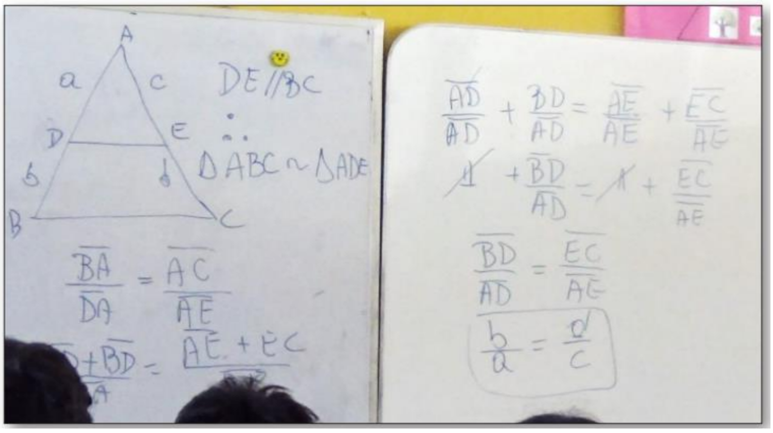
4.1. Case 1: Teaching of Thales' theorem

The session analyzed was implemented in a first-year secondary education course (ages 14-15); P1 declares that the objective is to understand and solve exercises on Thales' theorem (TT). During this session, the teacher writes and projects on the whiteboard the exercises that the students must solve. In the class, P1 establishes a moment to perform the demonstration of the theorem.

For this demonstration, P1 resorts to the representation of a figure to illustrate the similarity of two triangles (ABC and ADE) and establish proportional relationships drawing from the equality of segment ratios (Figure 3), that is, *BA/DA=AC/AE*, concluding that *BD/AD=EC/AE*, which is obtained from operating on the terms of the proportion as fractions. Substituting the aforementioned segments for *a, b, c*, and *d*, P1 concludes that *b/a=d/c*. The teacher signals that what has just been demonstrated is a special case of TT. Lastly, P1 stresses that the theorem is valid for parallel lines and that, without this hypothesis, the result is invalid.

Figure 3.

P1's work on the board



Source: Henríquez-Rivas et al. (2021, p. 134). https://doi.org/10.5565/rev/ensciencias.3210

From the MTSK perspective, it is posited beginning with the statement of the class objective that the purpose P1 attributes to teaching the theorem is its application in situations when calculating unknown data. In this manner, conceptual understanding is relegated to the understanding of the TT hypothesis as a necessary condition for its application; however, the exercises focus on the procedural. Thus, P1's knowledge of mathematical learning standards is evidenced (KMLS, expected level of conceptual and procedural development). Additionally, P1 relates the theorem with the similarity of plane figures using similar triangles (KoT: definitions, properties, and their foundations). For P1, the theorem is drawn from similarity, and its function is application to the calculation of unknown measurements. It should be mentioned that, in the current Chilean curriculum, TT is associated with homothecies and proportionality of segments. Therefore, the teacher's KMLS differs from the official curricular proposal and appears related to his own mathematical knowledge, in which TT is explained based on similarity (KoT) and without observing any manifestation of homothecy.

Likewise, P1 attributes importance to mathematical demonstrations. However, in the classroom he does not demonstrate TT in a general manner, but rather a particular case (Figure 3 above), considering it as a specific case and using it to justify what he is illustrating (KoT: definitions, properties, and their foundations). Here, the geometric work of TT is transferred to the algebraic domain, as the emphases is on the operations (algebraic) based on the similarity of triangles, establishing an auxiliary connection by developing certain proportional relationships that allow said operations to be carried out (KSM: auxiliary connection).

In terms of analyses from the MWS perspective, the initial activation via semiotic genesis is highlighted by the similar triangles drawn freehand and the process of visualization that entails the decomposition of Figure 3 (above), activating the [Sem-Ins] vertical plane. Then, the semiotic work entails the conversion of the figural register to the algebraic register, focused on algebraic operations, the use of properties of proportions, and the similarity of figures (as theoretical artefacts), which he uses to prove an equality between proportions. In this manner, the [Sem-Dis] vertical plan is privileged. Ultimately, P1 only presents the thesis of TT as a consequence of the similarity of triangles, only mentioning the hypothesis of parallelism.

In this case, an emphasis on algebraic treatments and operations can be observed stemming from an initial geometric representation. The work focused on processes of proof associated with TT is secondary, given that the teacher grants greater importance to the aforementioned treatments and operations. The discursive intention appears less relevant in the proof (equality between proportional relationships) or in the use of a given artefact for geometric construction. The theoretical referential component appears, but not explicitly, in his work. Likewise, P1's work seems influenced by his knowledge of the role of the demonstration as an argument underpinning the procedure associated with TT. This reflects knowledge regarding how demonstration is carried out in mathematics and an auxiliary connection between the similarity of figures and the demonstration of the theorem, evoking operations with fractions when operating with ratios in the algebraic register, which originate from geometric relations between the elements of TT.

In this case, a domain change is evident from the domain of origin (geometric) to the domain of resolution (algebraic), putting greater emphasis on the algebraic operations and without returning to the initial geometric work (Montoya & Vivier, 2014). Additionally, it can be noted how P1's knowledge of the practice of mathematics (KPM), in relation to the demonstration and the roles attributed to this in his classroom work, influence his proposal for teaching the theorem. He establishes TT as a consequence of similarity, which explains his knowledge of representations of the theorem, his interpretation, and the connections he establishes with fractions and proportionality.

4.2. Case 2: Teaching of the function

In the class analyzed, P2 includes a reminder about set language and the Cartesian plane, going on to demonstrate the definition of a function, the concepts of image and pre-image, and the different representations of the function. This displays his knowledge of the topic (KoT) and shows his KMLS regarding sequencing and which of these enhance the study of the function (set language and the Cartesian plane).

For the calculation of images and pre-images, P2 displays knowledge of two procedures that allow him to solve these tasks: evaluating the function and solving a linear equation, respectively, which is shown in the following classroom excerpt:

> P2: When you replace, now you're calculating your image. Considering the function, to determine the pre-image of 4, for example, what do I have to do? I want this [indicating the function] to give me result 4. This translates in solving an equation in place of groping around for the result. Do we know how to solve this here? This would be equivalent to 1-4=3x. And this would be -3=3x. So, what would x have to be?
> P2: […] Can we find a way to find the pre-image without playing a guessing game [testing values]?
> Students: Yes.
> P2: Yes. What we want is for this [image of the function] to give us 5. It turns into an equation. That's why we saw equations and inequalities before. (Espinoza, 2020, p. 152)

These procedures (replacing and solving) form part of P2's KoT. Although the teacher indicates that the pre-image can be obtained by testing values, he encourages the solving of equations as a general procedure for arriving at the answer for this task. This illustrates the procedural development that P2 expects from his students within his KMLS. Thus, the teacher establishes an auxiliary connection between the equation and the function (in his KSM), since the former represents concept different from the function the offers help in the work proposed.

P2 indicates that the students are familiar with equations (KMLS) and emphasizes their utility in working with functions. In particular, he links the equation $f(x)=0$ with the graphical representation of the function, relating the intersection between the graph of the function and the X-axis with the solution of said equation.

> P2: …[the function] is set to 0 only once. Were going to be able to relate the function with what we saw with first-order equations, because if I make the function equal to 0, what is the solution to that equation?
> Students: -3.
> P2: The solution to that equation is -3. The line, when it crosses the X-axis, that point I get there is the representation of the solution to the equation. (Espinoza, 2020, p. 153)

This excerpt shows P2's KoT regarding characteristics of the representation of the function and the solving of equations. Additionally, the affine function and the Euclidean line are related transversally (KSM) through sharing the same representation (straight line). This type of connection also reveals a domain change, from analysis (functions) to geometry (lines), where the basis for building the representation in question originates from the new domain that is put into play.

> P2: Remember what we had to do to graph this function [f(x)=x+3]? How many did we need as a minimum?
> Students: 2.
> P2: And, why two? […] What is the basis for needing at least two points?
> Student: To make a line.
> P2: Because we had said that… Two points have only one line passing through them, or one line has at least two points. With two values for X, I have enough. (Espinoza, 2020, p. 156)

Based on MWS, P2 can be observed to use previous knowledge to activate the component of the referential of the epistemological plane. The equation, meanwhile, acts as an operational tool to calculate pre-images, as it helps in the work with the function from another referential. This use permits a change in the work with functions, from analysis to work with equations in the algebraic domain. The domain change manifests when P2 signals that the work with the function and its pre-image "becomes an equation."

Additionally, the characteristics of the representation of the function and the solving of equations correspond to the referential, the representamen, and the artefacts of the epistemological plane. P2 seeks to activate semiotic genesis through interpretation and visualization of the solution for $f(x)=0$, with support from instrumental genesis through the use of the equation as an operational tool. This mathematical work also evinces a domain change from analysis to the algebraic y later to the geometric, when the teacher interprets the solution as an intersection point with the X-axis. In this geometric domain, P2 justifies the quantity of images that need to be calculated in order to construct the representation of the function, referencing the axiom of existence. This justification becomes an operational tool for the construction of the graphical representation of the function, activating with it instrumental genesis based on the theoretical referential in geometry and the representamen of the epistemological plane.

P2 selects and presents different types of tasks to his students to reach the objectives outlined: solving equations to determine pre-images of a function and graphically representing an affine or linear function. Such selection and organization of teaching reveals the structure of the suitable MWS and the knowledge involved in the design: KMT in terms of tasks and strategies, which is underpinned by his knowledge of the topic (KoT) and knowledge of connections (KSM), and supported by awareness of students' previous knowledge and what he hopes to achieve with them (KMLS).

Ultimately, the knowledge mobilized in the presentation of teaching of the calculation of images and pre-images of a function through the solving of equations allows for an explanation of the mathematical work that P2 proposes through the different geneses activated: instrumental (use of the equation), semiotic (representation of the function), and discursive (justifying the construction of the representation).

4.3. Case 3: Teaching of sequences

In the case of P3, an extract was selected from a task on sequences and the development presented by the professor as results expected from his students. The task (Figure 4) is analyzed combining elements of both MWS and MTSK.

Figure 4.

Presentation of task by P3 and its translation

PREGUNTA: Para $a > 0$ y $s_1 > 0$ se define la suceción (s_n) mediante la recurrencia:

$$s_{n+1} = \frac{{s_n}^2+a}{2s_n} \forall\, n > 1$$

a) Demuestre que la sucesión (s_n) es decreciente $\Leftrightarrow$ (s_n) es acotada inferiormente por $\sqrt{a}$.

Question: For $a > 0$ and $s_1 > 0$ the sequence (s_n) is defined by the following recurrence relation

$$s_{n+1} = \frac{s_n^2+a}{2s_n}, \quad \forall n > 1$$

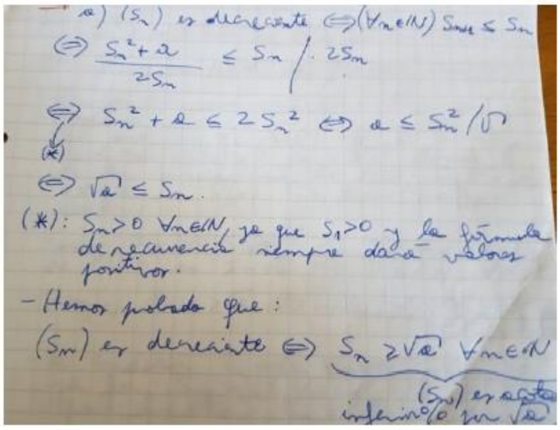
a) Prove that (s_n) is a decreasing sequence if and only if $(s_n)$is bounded below by a.

Source: Verdugo-Hernández et al. (2022, p. 135).
https://doi.org/10.5565/rev/ensciencias.3457%20

The first part reflects the referential and the semiotic aspects involved in the management of the succession, which can be observed both in the writing of the statement and the indication of delimitation by $\sqrt{a}$, since it invites the execution of an approach without indicating it directly. The latter displays P3's suitable MWS, and at the same time his personal MWS and a pronounced work from the referential toward the demonstration in the cognitive plane, which gives way to two interpretations. One the one hand, from MWS, it is viewed as a tool for obtaining the answer to the task. On the other hand, from MTSK, it is seen as a mathematical practice (KPM) to generate knowledge. Figure 5 shows, in detail, the development of the task that the professor carries out.

Figure 5.

Development of proposed task by P3



(1.a)	*a)* (s_n) is decreasing $\Leftrightarrow (\forall n \in \mathbb{N})\ s_{n+1} \le s_n$
(2.a)	$\Leftrightarrow \frac{s_n{}^2+a}{2s_n} \le s_n/2s_n$
(3.a)	$\Leftrightarrow s_n{}^2 + a \le 2s_n{}^2 \Leftrightarrow a \le 2s_n{}^2\ /\sqrt{\ }$
(4.a)	(*)
(5.a)	$\Leftrightarrow \sqrt{a} \le s_n$
(6.a)	(*):$s_n > 0\ \forall n \in \mathbb{N}$, since $s_1 > 0$ and the recurrence formula will always give positive values
(7.a)	- We have proven that
(8.a)	(s_n) is decreasing $\Leftrightarrow \underbrace{s_n \ge \sqrt{a}\ \forall n \in \mathbb{N}}_{s_n \text{ it is bounded below by } \sqrt{a}}$

Source: Verdugo-Hernández et al. (2022, p. 136). https://doi.org/10.5565/rev/ensciencias.3457%20

The solution presented is initiated by activating the referential of the sequence through its property of decreasing (1.a). This evidences P3's knowledge of said property as part of his KoT. From lines 2.a through 5.a (Figure 5 above), in terms of MWS, we can observe the use of inequality properties of real numbers and the order relationship as operational tools to demonstrate the decrease and bounding of the sequence. In turn, this corresponds to the professor's knowledge of the solving of inequalities (KoT of inequalities). In lines 3.a and 5.a, P3 applies his knowledge of the square root function (KoT) as an operator that helps solve the inequality and develop the proof requested, drawing on the order of the reals as an auxiliary connection and an operational tool. From lines 6.a through 8.a, it is proven that the lower bound of the sequence is zero through use of the principle of induction, utilized as a tool of natural numbers (KSM) that aids in the production of the demonstration (KPM).

Through this work, the P3's KPM is highlighted in different aspects. For example, from 2.a to 5.a, the role of equivalence is shown as symbology that influences the communication of the mathematical ideas, which is key in the structure of the demonstration. The use of the universal quantifier (1.a, 6.a, and 8.a) to indicate that the property is valid for all naturals also reflects knowledge of the role of the symbol and the structure and communication of this demonstration. Likewise, it is clear that the KPM organizes that which is manifested in the KoT when the knowledge of notations, procedures, and properties (KoT) allows the professor to structure the development of the demonstration.

In particular, knowledge of the use of notations and representations for sequences (KoT) is observable the in the procedure for solving inequalities and the use of the properties of the square root. This constitutes semiotic tools in the work that P3 expects, to which is added the role of symbology observed as KPM and as part of the semiotic tools involved. Table 4 synthesizes the MWS-MTSK relationships described.

Table 4.

Relationships between MWS tools and MTSK subdomains in the development of the task by P3

Tools of MWS	Subdomain of MTSK
Operational tool Principle of mathematical induction. Properties of inequalities of real numbers.	Auxiliary connection (KSM) Principle of mathematical induction as an auxiliary in the proof of equivalence. Order relation in real numbers to show monotonicity.
Operational tool Principle of mathematical induction.	Proof as practice in mathematics (KPM) Induction as a way of proving given propositions in terms of the natural numbers.
Operational tool Properties of inequalities of real numbers.	Procedures (KoT) Solving inequations.
Semiotic tool Use of Sn+1≤ Sn. Use of subindexes.	Representation registers (KoT) Use of symbology Sn+1≤ Sn. Notation and use of subindexes.
Semiotic tool Quantifier and equivalence.	Communication of mathematical ideas Role and use of quantifiers and equivalence to communicate proof.

Source: Verdugo-Hernández et al. (2022, p. 137). https://doi.org/10.5565/rev/ensciencias.3457%20

P3's answer points to his knowledge of the properties, justifications, and procedures necessarily to solve the task, which in turn correspond to different types of tools. For example, knowledge of the square root and the principle of induction allow the task to be made concrete; these can be identified as semiotic tools (Verdugo-Hernández et al., 2022). Based on this relation, domain changes are observed in this task through the use of operational tools. In 1.a, P3 changes domain from analysis to algebraic in 2.a through the inequality, in which he remains from 2.a through 5.a. In 6.a, there is a return to the analysis domain based on sequences, which connotes the inequality and induction as operational tools. P3 concludes the task in the analysis domain (sequences). It should be noted that the operational tool reveals a connection with objects from other domains and expresses KSM, but it does not necessarily imply that the work with that tool is undertaken in an origin distinct from that of the origin.

5. Discussion and conclusions

The results of this study aim to analyze how mathematical work is developed in different contexts of teaching and highlight the relations between the mathematical and pedagogical knowledge of teachers and the domain changes in said work. With the three cases presented, we seek to render an account of the implication of knowledge of the teacher in the interactions between different working spaces, as proposed by Gómez-Chacón et al. (2016), that emerge in the teaching process.

We begin by presenting the case of P1, with separate analyses using both models as undertaken by Climent et al. (2021) and Henríquez-Rivas et al. (2021), respectively. These studies include the teacher, and the mathematical task as focuses of study, opening the possibility of using both models for analysis simultaneously, as the teacher and task are seen as elements that permit connection between MWS and MTSK (Espinoza-Vásquez et al., 2022). Positing joint analysis associated with the same case entails a challenge for researchers, which manifests as identifying a common research problem in view of both theories. In the case of

P1, we present a demonstration task for TT during classroom teaching, which highlights discursive genesis in the teacher's mathematical work and attributes the role of verification and explanation to the demonstration (De Villiers, 1993), the latter as part of his knowledge of teaching the theorem and the demonstration itself (Delgado-Rebolledo & Zakaryan, 2019). Specifically, the task of demonstrating (or proving, in a wider sense) is reaffirmed as an element that allows joint analysis with MWS and MTSK for interpreting the teacher's practice in light of his knowledge.

In the case of P2, like P1, operational aspects of the topic stand out as learning expectations for the students. Both teachers propose solving equations to find pre-images and apply Thales' theorem, respectively, which is interpreted as specialized knowledge (of procedures and the depth of learning expected) that influence the formation of the suitable MWS (Espinoza-Vásquez et al., 2018). This is exhibited by the organization of teaching displayed in the classroom and, in particular, by the predominance of algebraic treatments.

The analysis of P2 also demonstrates the use of knowledge outside the conceptual network of the topic being taught. These elements are interpreted as tools and pieces of knowledge, using both models in combination (in the sense of Prediguer et al., 2008). The use of tools allows connections to be seen in the interior of the teacher's knowledge (in his KSM) and reflects the change of domain of origin of the ask to the domain of resolution, which in this study we seek to underscore. In particular, the appearance of operational tools in the teacher's work is linked to a change of referential and, with this, a change of the topic of work (Verdugo-Hernández et al., 2022). This is observed in the three cases presented.

The analysis of P3 focuses on the use of tools. Here it is evident that these are closely related to different types of knowledge and reflect connection between these areas of knowledge when the MTSK is incorporated in the interpretation of their us. According to Verdugo-Hernández et al. (2022), through tools (theoretical, semiotic, and operational) new connections are established between the theoretical models, for example, between KoT and the referential component through the use of definitions and properties of a topic, or between semiotic genesis and knowledge of the characteristics of representations of the objects of study. In this manner, the use of knowledge of the development of a task evinces the tools utilized by the teacher. The authors report that the operational tools are indicators of the use of objects from referentials different from the original topic in which the task was proposed, and the allow a view of knowledge of connections in the teacher's KSM; this occurs in the three cases illustrated.

In general terms, we observe that it is the use of different areas of knowledge (mathematical and pedagogical) that link the activity being carried out in terms of the development of a teaching task. For example, the knowledge (specialized) of definitions, theorems, or representations associated with a topic are mobilized and spur the emergence of related teaching strategies which, in turn, are supported by the selection of resources or relevant examples. This knowledge ensemble expresses an organization of learning situated in the suitable MWS of the teacher; the activation of these components is sought through actions based on the areas of knowledge previously described. The existence of natural connections between the MWS and MTSK theoretical perspectives, evidenced in their own formulations, and those which emerge from their joint application to a single dataset (in this study and previous studies), enable the refinement of analyses (Espinoza-Vásquez et al., 2022, and Verdugo-Hernández et al., 2022) and provide new interpretations of what each model obtains individually in terms of the teacher's practice.

In this manner, the three cases analyzed reaffirm the teacher and the task as elements that allow the joint interpretation of mathematical work with the models in question. Additionally, it is evident that the phenomenon of domain change is not an isolated matter within classroom practices, which opens a space for study applying MWS and MTSK simultaneously in this regard. This constitutes a contribution of this study, at both theoretical and methodological levels, for further research on the MWS-MTSK relationship, given that in said domain changes the specialized knowledge of the teacher intervenes and can be interpreted in its use as a component of the MWS model.

In this sense, the complementarity between MWS and MTSK facilitates new interpretations or deeper examinations of the teacher's practice. For example, while MWS characterizes the type of mathematical work presented, MTSK nourishes this characterization by identifying the type of knowledge mobilized and the relations that exist within it. In turn, the specialized knowledge identified through the MTSK model is interpreted, in terms of its use and specification of its components, as an enabler of the mathematical work presented in the MWS model. This double interpretation constitutes an analytical refinement and is posited as a benefit of the complementarity between MWS and MTSK. As stated by Espinoza-Vásquez et al. (2022), the connection between these models allows a deepening of the understanding of the teacher's practice, helps to comprehend its specialized character, and, based on said complementarity, also permits an understanding of the work proposed by the teacher in light of their own knowledge. Thus emerges the MWS-MTSK relationship, almost naturally, when the aforementioned mathematical practice is understood as mobilized by the teacher's specialized knowledge.

Finally, based on analyses emerging from both frameworks, we highlight certain aspects that can be addressed in future research. For example, we consider those aspects of a phenomenon which remain opaque working from one model (which we term blind spots), but which, from joint study, could contribute to enhancing research in the line of networking of theories that have been presented here. Likewise, exploring teachers' beliefs, as considered in the MTSK, could help structure the suitable MWS or elucidate gaps between the teaching that is planned and that which is actually implemented based on the suitable MWS (potential and actual). Furthermore, we consider that the joint application of MWS and MTSK as theoretical and methodological tools could extend to other mathematical topics or other teaching contexts, including pre-service teacher education, which opens new possibilities regarding the specialized knowledge of teacher educators and the referential MWS of the teacher.

6. References

Arias, M. (2000). La triangulación metodológica: sus principios, alcances y limitaciones. *Investigación y Educación en Enfermería*, *18*(1), 13-26. https://dialnet.unirioja.es/descarga/articulo/5331864.pdf

Bardín, L. (1996). *El análisis de contenido*. Akal Ediciones.

Carrillo, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, Ribeiro, M., & Muñoz-Catalán, M. (2018). The mathematics teacher's specialised knowledge (MTSK) model. *Research in Mathematics Education*, *20*(3), 236-253. https://doi.org/10.1080/14794802.2018.1479981

Carrillo, J., Contreras, L. C., Climent, N., Escudero-Avila, D., Flores-Medrano, E., & Montes, M. A. (Eds.). (2014). *Un marco teórico para el conocimiento especializado del profesor de matemáticas*. Universidad de Huelva Publicaciones.

Castela, C. (2021). *Reflexiones sobre la multiplicidad de las teorías en didáctica de las matemáticas*. HAL Open science. https://hal.science/hal-03199465

Climent, N., Espinoza-Vásquez, G., Carrillo, J., Henríquez-Rivas, C., & Ponce, R. (2021). Una lección sobre el teorema de Thales vista desde el conocimiento especializado del profesor. *Revista de Educación Matemática*, *33*(1), 98-124. http://doi.org/10.24844/EM3301.04

Creswell, J. W. (2014). *Research Design: Qualitative, Quantitative, and Mixed Methods Approaches*. Sage.

De Villiers, M. (1993). El papel y la función de la demostración en matemáticas. *Epsilon*, 26, 15-30. http://mzone.mweb.co.za/residents/profmd/proofb.pdf

Delgado-Rebolledo, R., & Zakaryan, D. (2019). Relationships between the knowledge of practices in mathematics and the pedagogical content knowledge of a mathematics lecturer. *International Journal of Science and Mathematics Education*, *18*(3), 567-587. https://doi.org/10.1007/s10763-019-09977-0

Espinoza-Vásquez, G., & Verdugo-Hernández, P. (2022). Las representaciones de la función durante la enseñanza. Una mirada desde el conocimiento especializado y el trabajo matemático del profesor. *Human Review*, *13*(6), 2-18. https://doi.org/10.37467/revhuman.v11.4082

Espinoza-Vásquez, G., Ribeiro, M., & Zakaryan, D. (2018). Avance en la comprensión de las relaciones entre el ETM idóneo y el MTSK del profesor. *Journal of Educational Research MENON*, 4, 146-161.

Espinoza-Vásquez, G., Verdugo-Hernández, P., Henríquez-Rivas, C., & Ponce, R. (2022). Avances en la relación entre MTSK y espacios de trabajo matemático. En J. Carrillo, M. A. Montes, & N. Climent (Eds.), *Investigación sobre conocimiento especializado del profesor de matemáticas (MTSK): 10 años de camino* (pp. 265-276). Dykinson S.L.

Espinoza, G. (2020). *Caracterización del conocimiento especializado del profesor de matemáticas de educación media sobre el concepto de función* [Tesis doctoral]. Pontificia Universidad Católica de Valparaíso.

Flores-González, M., & Montoya-Delgadillo, E. (2016). Artefacto y espacio de trabajo matemático en la multiplicación de números complejos. *Educación Matemática*, *28*(2), 85-117. https://doi.org/10.24844/em2802.04

Flores-Medrano, E., Montes, M., Carrillo, J., Contreras, L. C., Muñoz-Catalán, M., & Liñán, M. (2016). El papel del MTSK como modelo de conocimiento del profesor en las interrelaciones entre los espacios de trabajo matemático. *Boletín de Educación Matemática*, *30*(54), 204-221. https://doi.org/10.1590/1980-4415v30n54a10

Goetz, J. P., & LeCompte, M. D. (1988). *Etnografía y diseño cualitativo en investigación educativa*. Ediciones Morata.

Gómez-Chacón, I., Kuzniak, A., & Vivier, L. (2016). El rol del profesor desde la perspectiva de los espacios de trabajo matemático. *Boletín de Educación Matemática, Bolema*, *30*(54), 1-22. https://doi.org/10.1590/1980-4415v30n54a01

Henríquez Rivas, C., Kuzniak, A., & Masselin, B. (2022). The idoine or suitable MWS as an essential transition stage between personal and reference mathematical work. En A. Kuzniak, E. Montoya-Delgadillo, & P. Richard (Eds.), *Mathematical Work in Educational Context: The Perspective of the Theory of Mathematical Working Spaces* (pp. 121-146). Springer. https://doi.org/10.1007/978-3-030-90850-8_6

Henríquez-Rivas, C., Ponce, R., Carrillo, J., Climent, N., & Espinoza-Vásquez, G. (2021). Trabajo matemático de un profesor basado en tareas y ejemplos propuestos para la enseñanza. *Enseñanza de las Ciencias*, *39*(2), 123-142. https://doi.org/10.5565/rev/ensciencias.3210

Kuzniak, A., & Nechache, A. (2021). On forms of geometric work: a study with pre-service teachers based on the theory of mathematical working spaces. *Educational Studies in Mathematics*, 106, 271-289. https://doi.org/10.1007/s10649-020-10011-2

Kuzniak, A., Montoya-Delgadillo, E., & Richard, P. (2022). *Mathematical work in educational context: The perspective of the theory of mathematical working spaces*. Springer. https://doi.org/10.1007/978-3-030-90850-8

Kuzniak, A., Nechache, A., & Drouhard, J. P. (2016). Understanding the development of mathematical work in the context of the classroom. *The International Journal on Mathematics Education*, *48*(6), 861-874. http://doi.org/10.1007/s11858-016-0773-0

Lougrhan, J., Mulhall, P., & Berry, A. (2008). Exploring pedagogical content knowledge in science teacher education. *International Journal of Science Education*, *30*(10), 1301-1320. https://doi.org/10.1080/09500690802187009

Ministerio de Educación. (2015). *Bases curriculares 7° básico a 2° medio*. Ministerio de Educación de Chile.

Montoya, E., & Vivier, L. (2014). Les changements de domaine dans le cadre des espaces de travail mathématique. *Annales de Didactique et de Sciences Cognitives*, 19, 73-101. https://journals.openedition.org/adsc/2384

Prediger, S., Arzarello, F., Bosch, M., & Lenfant, A. (2008) (Eds.). Comparing, combining, coordinating - Networking strategies for connecting theoretical approaches. *ZDM - The International Journal on Mathematics Education*, *40*(2), 163-327. https://doi.org/10.1007/s11858-008-0093-0

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education*, *40*(2), 317-327. https://doi.org/10.1007/s11858-008-0090-3

Scheiner, T., Montes, M. A., Godino, J. D., Carrillo, J., & Pino-Fan, L. (2019). What makes mathematics teacher knowledge specialized? Offering alternative views. *International Journal of Science and Mathematics Education*, 17, 153-172. https://doi.org/10.1007/s10763-017-9859-6

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14. https://www.wcu.edu/WebFiles/PDFs/Shulman.pdf

Verdugo-Hernández, P. (2018). *Espacio de trabajo matemático del análisis: enseñanza de las sucesiones en los primeros años de universidad* [Tesis de Doctorado]. Pontificia Universidad Católica de Valparaíso.

Verdugo-Hernández, P., Espinoza-Vásquez, G., & Carrillo, J. (2022). Análisis de una tarea sobre sucesiones desde el uso de las herramientas y el conocimiento matemático del profesor. *Enseñanza de las Ciencias*, *40*(2), 125-145. https://doi.org/10.5565/rev/ensciencias.3457

Yin, R. K. (2009). *Case study research. Design and methods* (4ª ed.). SAGE Publications INC.

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